The Jacobi Bound Conjecture

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Slides available at https://dmzb.github.io/

$$\boxed{\dim \Sigma \leq J(\Sigma)}$$

Linear differential equations

Theorem (Euler? Cauchy?)

Let $P(t), G(t)$, and $R(t)$ be continuous functions.

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions to

$$P(t)\frac{d^2y}{dt^2} + G(t)\frac{dy}{dt} + R(t) = 0.$$

Then the general solution is given by

$$y(t) = c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 are arbitrary constants.

In other words, the <u>space of solutions</u> to a system of linear differential equations has **dimension** at most the <u>number of derivatives</u>.

The Jacobi Bound Conjecture (1865)[1]

DE INVESTIGANDO ORDINE SYSTEMATIS AEQUATIONUM DIFFERENTIALIUM VULGARIUM CUJUSCUNQUE.

"On investigating the order of a system of ordinary differential equations of any kind."

where $j_1, \cdots, j_n$ is a permutation of $1, \cdots, n$. He arrives at the conclusion that *the number of arbitrary constants in the solution of* (1) *does not exceed the greatest sum* (2).

The dimension of the manifold of solutions is at most the number of derivatives.

[1] posthumous

Ritt and Kolchin's Formalism of differential algebra

Definition

A **differential Ring** is pair (R, ∂), where
$\partial\colon R \to R$ is linear and satisfies the Leibniz rule.

E.g., $R = \mathbb{C}(t)$, or $R = \mathbb{C}(t, \cos t)$

Definition

The (differential) ring of **differential polynomials** is

$$k\{x_1, \ldots, x_n\} = k[x_1, x_1', x_1^{(2)}, \ldots, x_n, x_n', x_n^{(2)}, \ldots]$$

Definition

A **differential ideal** is an ideal closed under ∂, e.g.,

$$[f_1, \ldots, f_m] = \langle f_1, f_1', f_1^{(2)}, \ldots, f_m, f_m', f_m^{(2)}, \ldots \rangle$$

$x_1^2 + (\cos t)\, x_2' x_3' = 0$ is **OK**, but $\cos(x_1') = 0$ is **not**.

Example 1 from Ritt's Differential Algebra book (p. 30)

Example

$K = \mathbb{C}(t)$

$u = (y')^2 - 4y = 0$

Can solve directly :

$\boxed{f(t) = 0, \text{ or } (t+c)^2} \, c \in \mathbb{C}$

Note that $I = [u] = \langle u, u', u'', \ldots \rangle = \left[(y')^2 - 4y, \boxed{2y'y'' - 4y'}, \ldots \right]$

Since $u' = 2y'(y'' - 2)$ factors, $k\{y\}/I$ has two components:

$\mathfrak{p}_1 = [u, y'] \supset I$

Since $u = 0$, also $y = 0$, so $\boxed{k\{y\}/\mathfrak{p}_1 \cong k}$

$\mathfrak{p}_2 = [u, y'' - 2] \supset I$

$k\{y\}/\mathfrak{p}_2 \cong k[y, y']/\langle (y')^2 - 4y \rangle$, so $\boxed{k\{y\}/\mathfrak{p}_2 \cong k[y]}$

Definition

The **Jacobi number** of $(u_1, \ldots, u_n)$ is

$$J(u_1, u_2, \ldots, u_n) = \max_{\sigma \in S_n} \sum_{i=1}^{n} \operatorname{ord}^{\partial}_{x_i}(u_{\sigma(i)}).$$

NOTE: same number of equations as variables.

Example

$$\begin{aligned} u_1 &= x + x' + y^{(2)} + z^{(3)} \\ u_2 &= x' + y' + z' \\ u_3 &= x^{(2)} + y' + z^{(3)} \end{aligned}$$

This has $J = 6$ and **order matrix**

$$\left(\operatorname{ord}^{\partial}_{x_i} u_j\right) = \begin{pmatrix} 1 & 2 & \boxed{3} \\ 1 & \boxed{1} & 1 \\ \boxed{2} & 1 & 3 \end{pmatrix}$$

Formal statement of JBC

Definition

$$J(u_1, u_2, \ldots, u_n) = \max_{\sigma \in S_n} \sum_{i=1}^{n} \operatorname{ord}^{\partial}_{x_i}(u_{\sigma(i)}).$$

Conjecture (Jacobi Bound Conjecture (JBC))

Let Σ be defined by $I = [u_1, u_2, \ldots, u_n]$.

Let Σ_1 be an irreducible component of Σ.

*Suppose that Σ_1 has **finite Krull dimension**. Then*

$$\operatorname{Krull}(\Sigma_1) \leq J(u_1, u_2, \ldots, u_n).$$

Example 1 from Ritt's Differential Algebra book (p. 30)

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$\mathfrak{p}_2 = [u, y'' - 2] \supset I$

$k\{y\}/\mathfrak{p}_2 \cong k[y, y']/\langle (y')^2 - 4y \rangle$, so $\boxed{k\{y\}/\mathfrak{p}_2 \cong k[y]}$

Ritt (Annals, 1936)

Theorem (Ritt)

The linear case of JBC is true.

Theorem (Ritt)

The two variable case of JBC is true.

Main theorem

Theorem (Dupuy–Zureick-Brown)

Suppose that a component Σ_1 is finite dimensional and ***generically reduced****.*

Then JBC is true for Σ_1.

Theorem (Dupuy–Zureick-Brown)

The Dimension Conjecture implies JBC.

Remark

It was known that "strong" JBC implies the dimension conjecture.

Remark

We wrote a differential algebra Magma package (available on github.).

Proof sketch for first theorem

1. $\dim \Sigma_1 \leq \boxed{\dim T_\eta \Sigma_1 \leq J(T_\eta \Sigma_1)} \leq J(\Sigma_1)$
2. $T\Sigma_1$ is a differential variety, with **linear fibers**, thus handled by Ritt
3. $\dim T_\eta \Sigma_1 < \infty$ if Σ_1 is generically reduced

Problem: $\dim T_\eta \Sigma_1 = \infty$ if not generically reduced

$[y^2] \ni 2yy'$, $\boxed{yy'' + (y')^2}$, $y''(yy') + (y')^3$, so $\boxed{(y')^3 \in [y^2]}$, $(y'')^5 \in [y^2]$ etc.

Problem: Ritt's proof of the linear case is incomplete.

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Let us show that, (a) having been attended to, (b) can also be realized. We can arrange so that

$$a_{n1},\ a_{1n},\ a_{22},\ a_{33},\ \cdots,\ a_{n-1,\,n-1}$$

is a diagonal. If, when the subscripts of the L_i have been thus modified, $a_{11} \neq -\infty$, we have (b). Otherwise there is some $a_{j1}, j \neq n$, which is not $-\infty$. We interchange L_1 and L_j, then y_j and y_n, and (b) holds. That (c) can be realized is now evident.

"**Differential Gaussian elimination**" has problems

Example

$$\begin{aligned} u_1 &= x + \boxed{x'} + y^{(2)} + z^{(3)} \\ u_2 &= x' + y' + z' \\ u_3 &= \boxed{x^{(2)}} + y' + z' \end{aligned} \qquad \begin{pmatrix} 1 & 2 & \boxed{3} \\ 1 & \boxed{1} & 1 \\ \boxed{2} & 1 & 1 \end{pmatrix} \qquad J(u_1, u_2, u_3) = 6$$

$$\boxed{v_3 = u_3 - u_1'}$$

$$\begin{aligned} u_1 &= x + x' + y^{(2)} + z^{(3)} \\ u_2 &= x' + y' + z' \\ v_3 &= x' + y' - \boxed{y^{(3)}} + z' - \boxed{z^{(4)}} \end{aligned} \qquad \begin{pmatrix} 1 & \boxed{2} & 3 \\ \boxed{1} & 1 & 1 \\ 1 & 3 & \boxed{4} \end{pmatrix} \qquad J(u_1, u_2, u_3) = 7$$

Ritt's special "Jacobi number-preserving" forms

Max means maximum of column.

First form

$$\begin{pmatrix} \boxed{*} & * & * & * & * \\ \text{Max} & \boxed{*} & * & * & * \\ * & * & \boxed{*} & * & * \\ * & * & * & \boxed{*} & * \\ * & * & * & * & \boxed{*} \end{pmatrix}$$

Second form

$$\begin{pmatrix} * & * & * & * & \boxed{*} \\ * & \boxed{*} & * & * & * \\ * & * & \boxed{*} & * & * \\ * & * & * & \boxed{*} & * \\ \boxed{\text{Max}} & * & * & * & * \end{pmatrix} \longrightarrow \left(\begin{array}{cccc|c} \boxed{*} & * & * & * & * \\ * & \boxed{*} & * & * & * \\ * & * & \boxed{*} & * & * \\ * & * & * & \boxed{*} & * \\ \hline \text{Max} & * & * & * & * \end{array}\right)$$

Ritt Division

Definition

Pick a **ranking** $\prec$ on $K\{x_1, \ldots, x_n\}$ such that

1. $x_i^{(r)} \prec x_i^{(r+1)}$, and
2. if $x_i^{(r)} \prec x_j^{(s)}$, then $x_i^{(r+1)} \prec x_j^{(s+1)}$.

$$sf = Q_1(g_1) + \cdots + Q_r(g_r) + r$$

Example

Let $f = x' + \boxed{y'''}$ and $g = x^2 + \boxed{y''}\,x' + t$ in $\mathbb{C}(t)\{x, y\}$. Then

$$sf = Qg + r$$

where

$$s = -(x')^2, \quad \boxed{Q = x'' - x'\partial}, \quad r = -x^2x'' + -tx'' + -(x')^3 + 2x(x')^2 + x'.$$

Ideal membership

Definition

$g_1, \ldots, g_r \in \mathfrak{p}$ is a **Characteristic Set** if the **ideal membership test** holds, i.e., if we write

$$sf = Q_1(g_1) + \cdots + Q_r(g_r) + r,$$

then

$$f \in \mathfrak{p} \text{ if and only if } r = 0$$

Basically a slightly worse Gröbner basis: it isn't true that

$$\mathfrak{p} = [g_1, \ldots, g_r]$$

but rather

$$\mathfrak{p} = \mathrm{sat}_S[g_1, \ldots, g_r]$$

where S is the set of "separants" and "initials",
i.e., the terms we "multiply by on the left" when dividing.

The Dimension Conjecture in $\operatorname{char} k = 0$

Differential dimension of A is the largest i s.t. $\exists k\{t_1, \ldots, t_i\} \hookrightarrow A$.

Conjecture (Dimension Conjecture (Cohn 1983, Ritt 1950))

Let $u_1, \ldots, u_m \in k\{x_1, \ldots, x_n\}$ *and let*

$$\Sigma = \operatorname{Spec} k\{x_1, \ldots, x_n\}/[u_1, \ldots, u_m].$$

Suppose that $\boxed{m < n}$. *Then*

$$n - m \leq \dim_K^{\partial}(\Sigma).$$

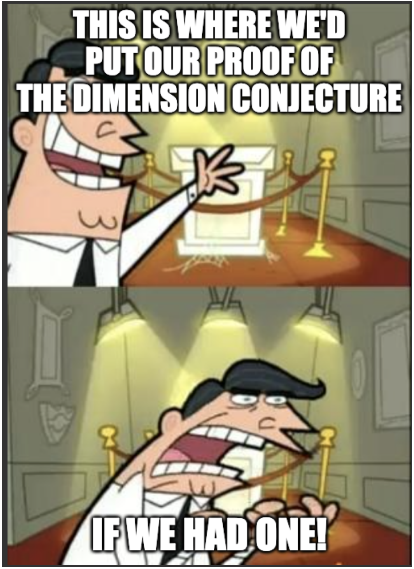
Remark

$m = 1$ *is the differential version of Krull's principal ideal theorem.*

Main issue:

When you add **one** equation, you are also adding all of its derivatives

So did we prove the Dimension Conjecture?



We're writing a "differential Atiyah–Macdonald".

About half generalizes cleanly.

Problem: infinite dimensionality prevents induction

Let $[u_1, u_2, \ldots, u_n] \supset \mathfrak{p}$ be a minimal prime.

Ritt divide:

$$su_1 = Qu_2 + v_1$$

and **replace**

$$[v_1, u_2, \ldots, u_n] \supset \mathfrak{p}' \supset \mathfrak{p},$$

Problems:

1. $\mathfrak{p}$ may no longer be minimal.
2. The new minimal prime $\mathfrak{p}'$ might be infinite dimensional.
3. (And thus we can't apply a known case of JBC.)

If s is **invertible** on $\mathfrak{p}$ (i.e., $s \notin \mathfrak{p}$), then

$$\mathfrak{p}' = \mathfrak{p}$$

Otherwise, $s \in \mathfrak{p}$, and we can also try replacing u_1 by s.

But this usually gives the same problem

Very few ways to prove finite dimensionality

Example

$k\{y\}/[f(\bar{y})]$ is “visibly” finite dimensional

“Cylinders”

$k\{x_1,\ldots,x_n\}/[f_1(\bar{x}_1),\ldots,f_n(\bar{x}_n)]$ is finite dimensional (by induction).

Example

If $f_1(x_1),\ldots,f_n(x_n) \in I$, then

$$k\{x_1,\ldots,x_n\}/[f_1(\bar{x}_1),\ldots,f_n(\bar{x}_n)] \to k\{x_1,\ldots,x_n\}/I$$

So $k\{x_1,\ldots,x_n\}/I$ is finite dimensional.

Elimination ideals $I \cap k\{x_i\}$ give a “projection π onto coordinate axis”.

When $I \cap k[x_i]$ contains a nonconstant polynomial, $\pi(X)$ is finite.

Degenerations and Ritt Pencils

Ritt's idea in the 2-variable case: consider $s, t \in \mathfrak{p}$ and form a family

$$[s + \mu t, u_2, \ldots, u_n] \supset \mathfrak{p}'_\mu \supset \mathfrak{p}.$$

When $n = 2$, everything is "so small" that he can "force" some fibers into a cylinder.

Remark

In higher dimensions this leads one back to the dimension conjecture.

A hint for why: we originally had n variables and n equations, but in the family we have $n + 1$ variables and n equations.

Also, we have access to a lot more commutative algebra than Ritt did (e.g., Characteristic Sets).

Thank you!

$$\dim \Sigma_1 \leq J(\Sigma_1)$$

$$\dim \Sigma_1 \leq \boxed{\dim T_\eta \Sigma_1 \leq J(T_\eta \Sigma_1)} \leq J(\Sigma_1)$$

$$\dim_K^\partial(\Sigma) \geq n - m.$$