ℓ-adic images of Galois for elliptic curves over $\mathbb{Q}$

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with Jeremy Rouse and Andrew V. Sutherland
and an appendix with John Voight

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Galois Representations

$$\begin{aligned} \mathbb{Q} &\subset K \subset \overline{\mathbb{Q}} \\ G_K &:= \mathrm{Aut}(\overline{K}/K) \\ E[n](\overline{K}) &\cong (\mathbb{Z}/n\mathbb{Z})^2 \end{aligned}$$

$$\begin{aligned} \rho_{E,n}: \quad & G_K \to \mathrm{Aut}\, E[n] & \cong \quad & \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \\ \rho_{E,\ell^\infty}: \quad & G_K \to \mathrm{GL}_2(\mathbb{Z}_\ell) & = \quad & \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \rho_E: \quad & G_K \to \mathrm{GL}_2(\widehat{\mathbb{Z}}) & = \quad & \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \end{aligned}$$

Serre's Open Image Theorem

Theorem (Serre, 1972)

Let E be an elliptic curve over K without CM. The image

$$\rho_E(G_K) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

of ρ_E is open.

Note:

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_\ell)$$

Thus ρ_{E,ℓ^∞} is surjective for all but finitely many ℓ.

For CM curves, see Lozano-Robledo's paper and work by Bourdon, Clark, and Pollack.

Image of Galois

$$\rho_{E,n}\colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}} \left\{ \begin{array}{l} \overline{\mathbb{Q}} \\ \big| \\ \left. \begin{array}{l} \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ \big| \\ \mathbb{Q} \end{array} \right\} H(n) \end{array} \right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Mazur's Program B

As presented at Modular functions in one variable V in Bonn

Theorem 1 also fits into a general program:

B. <u>Given a number field</u> K <u>and a subgroup</u> H <u>of</u> $GL_2\hat{\mathbb{Z}} = \prod_p GL_2\mathbb{Z}_p$ <u>classify all elliptic curves</u> $E_{/K}$ <u>whose associated Galois representation on torsion points maps</u> $\mathrm{Gal}(\overline{K}/K)$ <u>into</u> $H \subset GL_2\hat{\mathbb{Z}}$.

Mazur - Rational points on modular curves (1977)

Example - torsion on an elliptic curve

If E has a K-rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= \quad P \\ \sigma(Q) &= a_\sigma P \; + \; b_\sigma Q \end{aligned}$$

Example - Isogenies

If E has a K-rational, **cyclic isogeny** $\phi\colon E \to E'$ with $\ker\phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= a_\sigma P \\ \sigma(Q) &= b_\sigma P + c_\sigma Q \end{aligned}$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\mathrm{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\mathrm{sp}}$ and $H(n) \not\subset C_{\mathrm{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- whose kernels intersect trivially,
- neither of which is defined over K,
- but which are both defined over some quadratic extension of K,
- and which are Galois conjugate.

Example - other maximal subgroups

$\mathbb{F}_{p^2}^*$ acts on $\mathbb{F}_{p^2} \cong \mathbb{F}_p \times \mathbb{F}_p$

Normalizer of a non-split Cartan:

$$C_{\mathsf{ns}} = \mathrm{im}\left(\mathbb{F}_{p^2}^* \to \mathrm{GL}_2(\mathbb{F}_p)\right) \subset N_{\mathsf{ns}}$$

$H(n) \subset N_{\mathsf{ns}}$ and $H(n) \not\subset C_{\mathsf{ns}}$ iff

E admits a “necklace” (Rebolledo, Wuthrich)

Image of Galois

$$\rho_{E,n} \colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}}\left\{\begin{array}{l}\overline{\mathbb{Q}}\\ \Big| \\ \left.\begin{array}{l}\overline{\mathbb{Q}}^{\ker\rho_{E,n}} = \mathbb{Q}(E[n]) \\ \Big| \\ \mathbb{Q}\end{array}\right\} H(n)\end{array}\right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q\rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ \ni \ (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Let $\Gamma(N) \subset H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$. The minimal such N is the **level** of H.

Definition

$X_H := X(N)/H(N)$ (where $H(N)$ is the image of H in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$)

$X_H(K) \ni (E/K, \iota) \Leftrightarrow \rho_{E,N}(G_K) \subset H(N)$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

1. $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
2. if $-I \in H$.

Rational Points on modular curves

Mazur's program B

Compute $X_H(\mathbb{Q})$ for all H.

Remark

- *Sometimes* $X_H \cong \mathbb{P}^1$ *or elliptic with rank* $X_H(\mathbb{Q}) > 0$.
- *Some* X_H *have* ***exceptional*** *points (i.e, non-cusp non-CM points).*
- *Can compute* $g(X_H)$ *group theoretically (via Riemann–Hurwitz).*

Fact

$$g(X_H), \gamma(X_H) \to \infty \text{ as } \left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H\right] \to \infty.$$

(Serre) Sample subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$$\begin{array}{ccccccc}
\ker\phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_2 = 3 \\
 & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \\
I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_1 = 4 \\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow & & \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & &
\end{array}$$

$\chi\colon\ \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3$.

$\chi = \mathsf{sgn} \times \det$

$H(8) := \chi^{-1}(G)$, $G \subset \mathbb{F}_2^3$.

A typical subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$$
\begin{array}{ccccccc}
\ker\phi_4 & \subset & H(32) & \subset & \mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_4 = 4 \\
 & & \big\downarrow{\scriptstyle \phi_4} & & \big\downarrow & & \\
\ker\phi_3 & \subset & H(16) & \subset & \mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_3 = 3 \\
 & & \big\downarrow{\scriptstyle \phi_3} & & \big\downarrow & & \\
\ker\phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_2 = 2 \\
 & & \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow & & \\
\ker\phi_1 & \subset & H(4) & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_1 = 3 \\
 & & \big\downarrow{\scriptstyle \phi_1} & & \big\downarrow & & \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & &
\end{array}
$$

Non-abelian entanglements

There exists a surjection $\theta\colon \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

$$
\begin{tikzcd}
 & H(6) := \Gamma_\theta \arrow[ddl, two heads] \arrow[ddr, two heads] \arrow[d, phantom, "\cap"] & \\
 & \mathrm{GL}_2(\mathbb{Z}/6\mathbb{Z}) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & & \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})
\end{tikzcd}
$$

Brau–Jones

$$\operatorname{im}\rho_{E,6} \subset H(6) \Leftrightarrow j(E) = 2^{10}3^3t^3(1-4t^3) \Rightarrow K(E[2]) \subset K(E[3])$$

$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1)$$

Main conjecture

Conjecture (Serre)

Let E be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.

In other words, conjecturally, $\rho_{E,\ell^\infty} = \mathrm{GL}_2(\mathbb{Z}_\ell)$ for $\ell > 37$.

“Vertical” image conjecture

Conjecture

There exists a constant N such that for every $E/\mathbb{Q}$ without CM

$$\left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_E(G_{\mathbb{Q}})\right] \leq N.$$

Remark

This follows from the “$\ell > 37$” conjecture.

Problem

Assume the “$\ell > 37$” conjecture and compute N.

Labeling subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ up to conjugacy

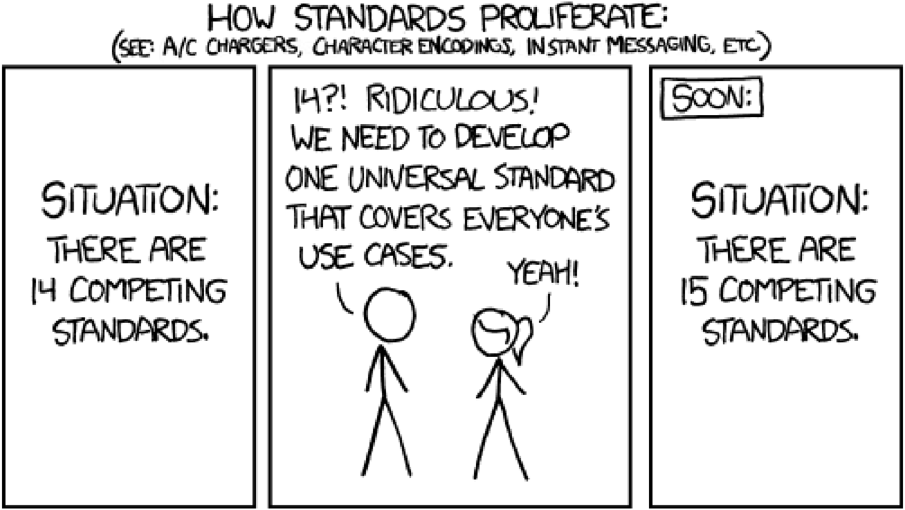
Definition

When $\det(H) = \widehat{\mathbb{Z}}^\times$ these labels have the form `N.i.g.n`, where N is the level, i is the index, g is the genus, and n is a tiebreaker given by ordering the subgroups of $\mathrm{GL}_2(N)$.

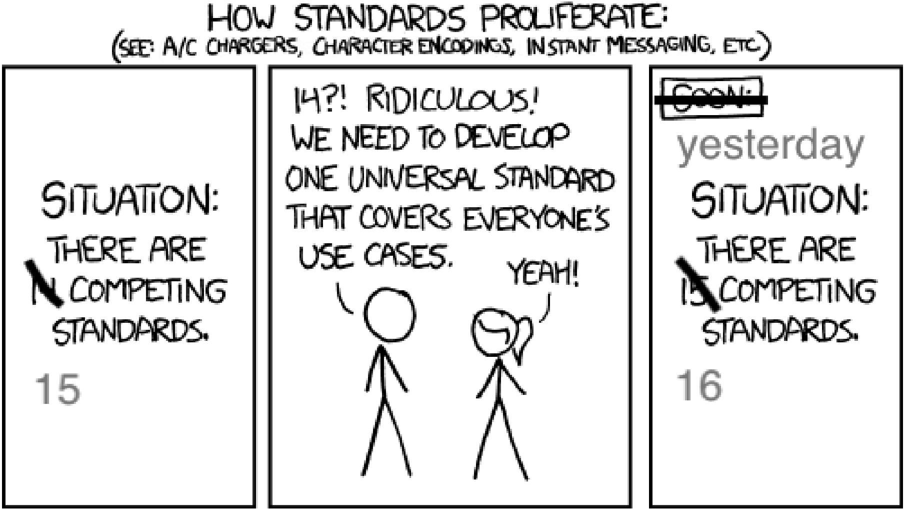
Example

- The Borel subgroup $B(13)$ has label `13.14.0.1`.
- The normalizer of the split Cartan $N_{\mathrm{sp}}(13)$ has label `13.91.3.1`.
- The normalizer of the nonsplit Cartan $N_{\mathrm{ns}}(13)$ has label `13.78.3.1`.
- The maximal S_4 exceptional group $S_4(13)$ has label `13.91.3.2`.

Obligatory XKCD cartoon



Obligatory XKCD cartoon



Main Theorem

Definition

A point $(E, \iota) \in X_H(K)$ is exceptional if $X_H(K)$ is finite and $\operatorname{End} E = \mathbb{Z}$.

Theorem (Rouse–Sutherland–ZB 2021)

Let ℓ prime, $E/\mathbb{Q}$ be a non-CM elliptic curve, and $H = \rho_{E,\ell^\infty}(G_\mathbb{Q})$. Then exactly one of the following is true:

1. *$X_H(\mathbb{Q})$ is infinite and H is listed in (Sutherland–Zywina 2017);*
2. *X_H has a rational exceptional point listed in Table 1;*
3. *$H \leq N_{\mathrm{ns}}(3^3), N_{\mathrm{ns}}(5^2), N_{\mathrm{ns}}(7^2), N_{\mathrm{ns}}(11^2)$, or $N_{\mathrm{ns}}(\ell)$ for some $\ell > 13$;*
4. *H is a subgroup of `49.179.9.1` or `49.196.9.1`.*

We conjecture that cases (3) and (4) never occur.

If they do, the exceptional points have **extraordinarily** large heights (e.g. $10^{10^{200}}$ for $X_{\mathrm{ns}}^+(11^2)(\mathbb{Q})$).

label	level	notes	j-invariants/models of exceptional points
16.64.2.1	2^4	$N_{\mathrm{ns}}(16)$	$-2^{18}\cdot 3\cdot 5^3\cdot 13^3\cdot 41^3\cdot 107^3/17^{16}$, $-2^{21}\cdot 3^3\cdot 5^3\cdot 7\cdot 13^3\cdot 23^3\cdot 41^3\cdot 17$
16.96.3.335	2^4	$H(4)\subsetneq N_{\mathrm{sp}}(4)$	$257^3/2^8$
16.96.3.343	2^4	$H(4)\subsetneq N_{\mathrm{sp}}(4)$	$17^3\cdot 241^3/2^4$
16.96.3.346	2^4	$H(4)\subsetneq N_{\mathrm{sp}}(4)$	$2^4\cdot 17^3$
16.96.3.338	2^4	$H(4)\subsetneq N_{\mathrm{sp}}(4)$	2^{11}
32.96.3.230	2^5	$H(4)\subsetneq N_{\mathrm{sp}}(4)$	$-3^3\cdot 5^3\cdot 47^3\cdot 1217^3/(2^8\cdot 31^8)$
32.96.3.82	2^5	$H(8)\subsetneq N_{\mathrm{sp}}(8)$	$3^3\cdot 5^6\cdot 13^3\cdot 23^3\cdot 41^3/(2^{16}\cdot 31^4)$
25.50.2.1	5^2	$H(5)=N_{\mathrm{ns}}(5)$	$2^4\cdot 3^2\cdot 5^7\cdot 23^3$
25.75.2.1	5^2	$H(5)=N_{\mathrm{sp}}(5)$	$2^{12}\cdot 3^3\cdot 5^7\cdot 29^3/7^5$
7.56.1.2	7	$\subsetneq N_{\mathrm{ns}}(7)$	$3^3\cdot 5\cdot 7^5/2^7$
7.112.1.2	7	$-I\not\in H$	$y^2+xy+y=x^3-x^2-2680x-50053$, $y^2+xy+y=x^3-x^2-1313$
11.60.1.3	11	$\subsetneq B(11)$	$-11\cdot 131^3$
11.120.1.8	11	$-I\not\in H$	$y^2+xy+y=x^3+x^2-30x-76$
11.120.1.9	11	$-I\not\in H$	$y^2+xy=x^3+x^2-2x-7$
11.60.1.4	11	$\subsetneq B(11)$	-11^2
11.120.1.3	11	$-I\not\in H$	$y^2+xy=x^3+x^2-3632x+82757$
11.120.1.4	11	$-I\not\in H$	$y^2+xy+y=x^3+x^2-305x+7888$
13.91.3.2	13	$S_4(13)$	$2^4\cdot 5\cdot 13^4\cdot 17^3/3^{13}$, $-2^{12}\cdot 5^3\cdot 11\cdot 13^4/3^{13}$, $2^{18}\cdot 3^3\cdot 13^4\cdot 127^3\cdot 139^3\cdot 157^3\cdot 2$
17.72.1.2	17	$\subsetneq B(17)$	$-17\cdot 373^3/2^{17}$
17.72.1.4	17	$\subsetneq B(17)$	$-17^2\cdot 101^3/2$
37.114.4.1	37	$\subsetneq B(37)$	$-7\cdot 11^3$
37.114.4.2	37	$\subsetneq B(37)$	$-7\cdot 137^3\cdot 2083^3$

Table 1. All known exceptional groups, j-invariants, and points of prime power level.

UNSOLVED mysteries

Arithmetically maximal level ℓ^n groups with $\ell \leq 13$ with $X_H(\mathbb{Q})$ unknown.

label	level	group	genus
27.243.12.1	3^3	$N_{\mathrm{ns}}(3^3)$	12
25.250.14.1	5^2	$N_{\mathrm{ns}}(5^2)$	14
49.1029.69.1	7^2	$N_{\mathrm{ns}}(7^2)$	69
49.147.9.1	7^2	$\left\langle \begin{pmatrix} 16 & 6 \\ 20 & 45 \end{pmatrix}, \begin{pmatrix} 20 & 17 \\ 40 & 36 \end{pmatrix} \right\rangle$	9
49.196.9.1	7^2	$\left\langle \begin{pmatrix} 42 & 3 \\ 16 & 31 \end{pmatrix}, \begin{pmatrix} 16 & 23 \\ 8 & 47 \end{pmatrix} \right\rangle$	9
121.6655.511.1	11^2	$N_{\mathrm{ns}}(11^2)$	511

Each has rank $=$ genus, rational CM points, no rational cusps, and no known exceptional points.

Summary of ℓ-adic images of Galois for non-CM $E/\mathbb{Q}$.

ℓ	2	3^*	5^*	7^*	11^*	13	17^*	37^*	other*
subgroups	1208	47	25	17	8	12	3	3	1
exceptional	7	0	2	2	6	1	2	2	0
unexceptional	1201	47	23	15	2	11	1	1	1
max level	32	27	25	7	11	13	17	37	1
max index	96	72	120	112	120	91	72	114	1
max genus	3	0	2	1	1	3	1	4	0

Summary of $H \leq \mathrm{GL}_2(\mathbb{Z}_\ell)$ which occur as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for some non-CM $E/\mathbb{Q}$.

Starred primes depend on the conjecture that cases (3) and (4) of our theorem do not occur.

In particular, we conjecture that there are $1207, 46, 24, 16, 7, 11, 2, 2$ proper subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ that arise as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for non-CM $E/\mathbb{Q}$ for $\ell = 2, 3, 5, 7, 11, 13, 17, 37$ and none for any other ℓ.

Applications

Theorem (R. Jones, Rouse, ZB)

1. ***Arithmetic dynamics***: *let* $P \in E(\mathbb{Q})$.
2. *How often is the order of* $\widetilde{P} \in E(\mathbb{F}_p)$ *odd?*
3. *Answer depends on* $\rho_{E,2^\infty}(G_{\mathbb{Q}})$.
4. *Examples:* $11/21$ *(generic),* $121/168$ *(maximal),* $1/28$ *(minimal)*

Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)

Classification of $E(\mathbb{Q}(3^\infty))_{tors}$

Theorem (Gonzalez-Jimenez, Lozanon-Robledo)

Classify $E/\mathbb{Q}$ *with* $\rho_{E,N}(G_{\mathbb{Q}})$ *abelian.*

Theorem (Rouse–Sutherland–ZB)

Improved algorithms for computing $\rho_{E,n}(G_{\mathbb{Q}})$.

Arithmetically maximal groups

Definition

We say that an open subgroup $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is arithmetically maximal if

1. $\det(H) = \widehat{\mathbb{Z}}^\times$ (necessary for $\mathbb{Q}$-points),
2. a conjugate of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ lies in H (necessary for $\mathbb{R}$-points),
3. $j(X_H(\mathbb{Q}))$ is finite but $j(X_{H'}(\mathbb{Q}))$ is infinite for $H \subsetneq H' \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

Arithmetically maximal groups H arise as maximal subgroups of an H' with $X_{H'}(\mathbb{Q})$ infinite.

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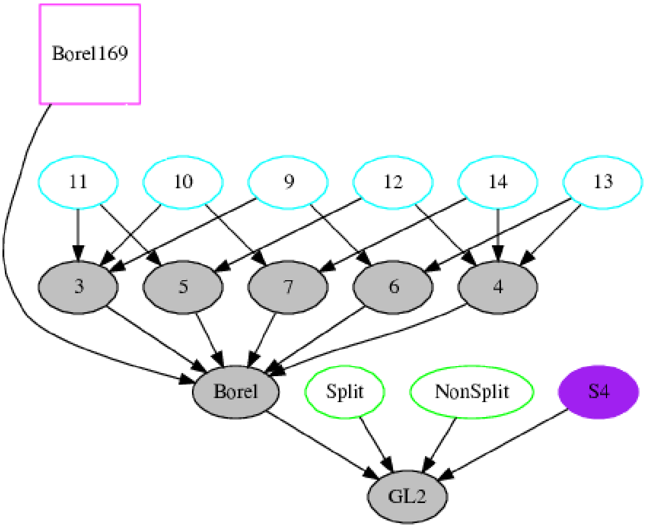
Arithmetically maximal groups H arise as maximal subgroups of an H' with $X_{H'}(\mathbb{Q})$ infinite.

Theorem (Sutherland–Zywina 2017)

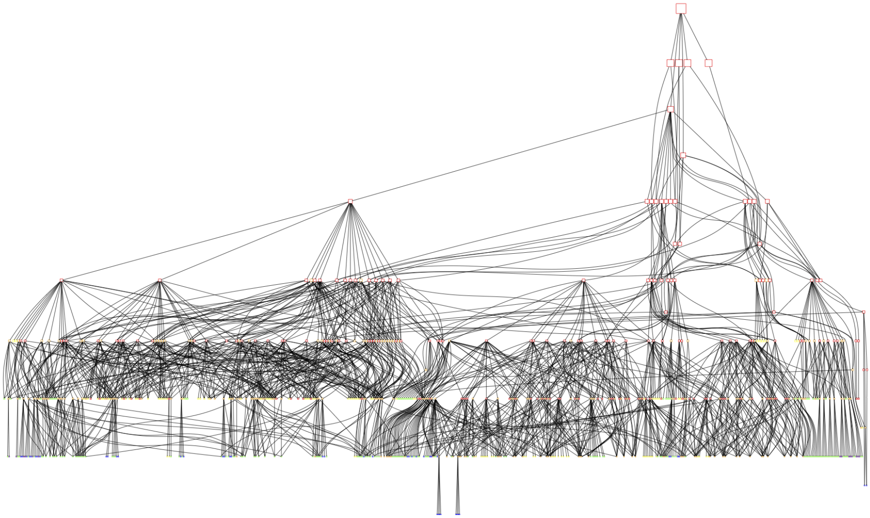
For $\ell = 2, 3, 5, 7, 11, 13$ there are $1208, 47, 23, 15, 2, 11$ subgroups $H \le \mathrm{GL}_2(\widehat{\mathbb{Z}})$ of ℓ-power level with $X_H(\mathbb{Q})$ infinite, and only $H = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for $\ell > 13$.

This allows us to compute explicit upper bounds on the level and index of arithmetically maximal subgroup of prime power level ℓ and we can then exhaustively enumerate them.

Subgroups of $\mathrm{GL}_2(\mathbb{Z}_{13})$



Subgroups of $\mathrm{GL}_2(\mathbb{Z}_2)$



Steps of the proof

1. Compute the set $\mathcal{S}$ of **arithmetically maximal** subgroups of ℓ-power level for $\ell \leq 37$ (for all $\ell > 37$ we already know $N_{\mathrm{ns}}(\ell)$ is the only possible exceptional group).

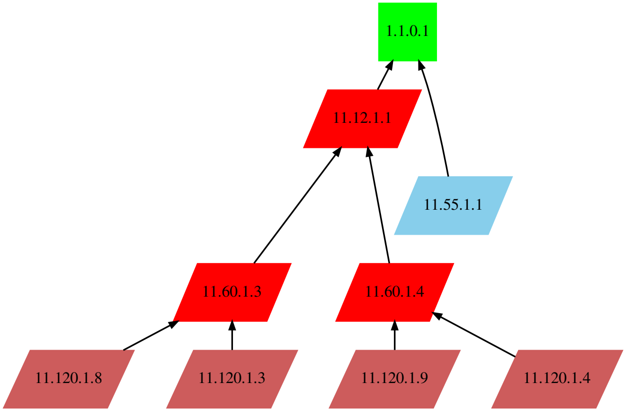
2. For $H \in \mathcal{S}$ check for **local obstructions** and compute the **isogeny decomposition** of the Jacobian of X_H and the analytic ranks of all its simple factors.

3. For $H \in \mathcal{S}$ **compute equations** for X_H and $j_H\colon X_H \to X(1)$ (if needed). In several cases we can prove $X_H(\mathbb{Q})$ is empty without a model for X_H.

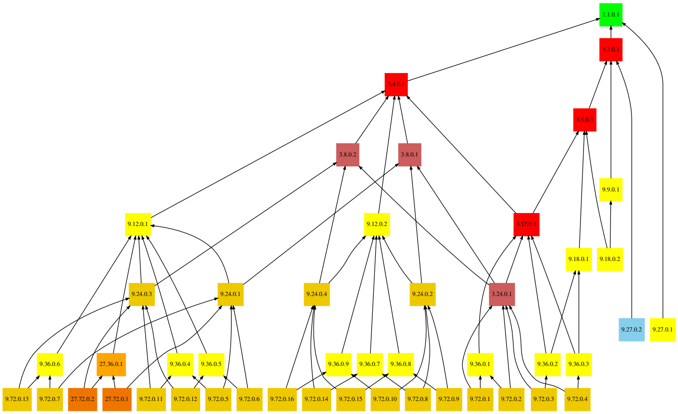
4. For $H \in \mathcal{S}$ with $-I \in H$ **determine the rational points** in $X_H(\mathbb{Q})$ (if possible). In several cases we are able to exploit recent progress by others ($\ell = 13$ for example).

5. For $H \in \mathcal{S}$ with $-I \notin H$ **compute equations** for the universal curve $\mathcal{E} \to U$, where $U \subseteq X_H$ is the locus with $j(P) \neq 0, 1728, \infty$.

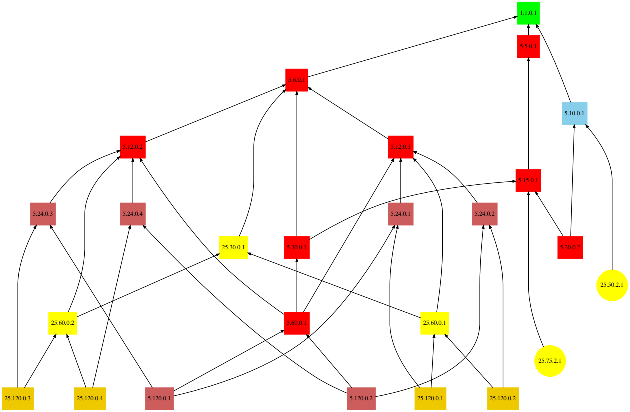
Subgroups of $\mathrm{GL}_2(\mathbb{Z}_{11})$



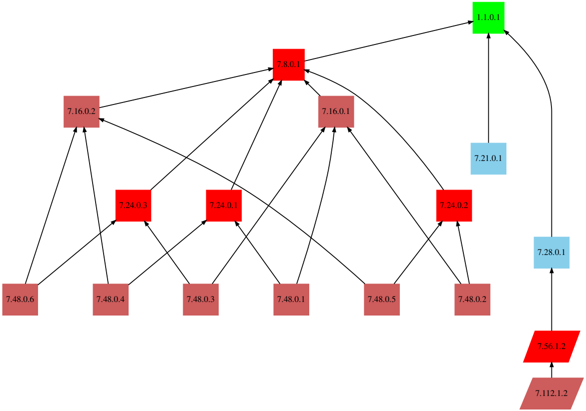
Subgroups of $\mathrm{GL}_2(\mathbb{Z}_3)$



Subgroups of $\mathrm{GL}_2(\mathbb{Z}_5)$



Subgroups of $\mathrm{GL}_2(\mathbb{Z}_7)$



Finding Equations for X_H – Basic idea

1. The canoncial map $C \hookrightarrow \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$.
2. For a general curve, this is an embedding, and the relations are quadratic.
3. For a modular curve,

$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$

given by

$$f(z) \mapsto f(z)\, dz^{\otimes k/2}.$$

Equations – Example: $X_1(17) \subset \mathbb{P}^4$

Cusp forms

$$q - 11q^5 + 10q^7 + O(q^8)$$
$$q^2 - 7q^5 + 6q^7 + O(q^8)$$
$$q^3 - 4q^5 + 2q^7 + O(q^8)$$
$$q^4 - 2q^5 + O(q^8)$$
$$q^6 - 3q^7 + O(q^8)$$

$$xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0$$
$$xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0$$
$$2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0$$

Computing models of modular curves

- We introduce a variety of improvements and tricks to compute models of various X_H.
- See Rouse's VaNTAGe talk for more details and interesting examples.
- To compute $j_H\colon X_H \to X(1)$ we represent E_4 and E_6 as ratios of elements of the canonical ring.
- We show that E_4 is a rational function of an element of weight k and weight $k-4$ if

$$k \geq \frac{2e_\infty + e_2 + e_3 + 5g - 4}{2(g-1)}$$

- We used this method to compute canonical models for many curves of large genus.
- See Assaf's recent paper and Zywina's BIRS talk for other efficient approaches.

Explicit methods: highlight reel

- Local methods
- Chabauty and Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- *Equationless étale descent via group theory*
- *New techniques for computing* Aut C

- *Nonabelian Chabauty*
- **“Equationless” local methods** and **Mordell–Weil sieve**
- **Greenberg Transforms** (and big computations)
- **Novel variants of existing techniques**
- **Modularity of isogeny factors of** J_H (w/ Voight)

Computing $X_H(\mathbb{F}_p)$ "via moduli"

Idea: one can compute $\#X_1(N)(\mathbb{F}_p)$ by enumerating elliptic curves over $\mathbb{F}_p$, then computing their N torsion subgroups.

Deligne–Rapoport 1973

The modular curves X_H and Y_H are coarse spaces for the stacks $\mathcal{M}_H$ and $\mathcal{M}_H^0$ that parameterize elliptic curves E with H-level structure, by which we mean an equivalence class $[\iota]_H$ of isomorphisms $\iota\colon E[N] \to \mathbb{Z}(N)^2$, where $\iota \sim \iota'$ if $\iota = h \circ \iota'$ for some $h \in H$.

- $Y_H(\bar{k}) = \{(j(E), \alpha) : \alpha = Hg\mathcal{A}_E\}$ with $\mathcal{A}_E := \{\varphi_N : \varphi \in \mathrm{Aut}(E_{\bar{k}})\}$, and $Y_H(k) = Y_H(\bar{k})^{G_k}$.
- $X_H^\infty(k) = \{\alpha \in H\backslash \mathrm{GL}_2(N)/U(N) : \alpha^{\chi_N(G_K)} = \alpha\}$ where $U(N) := \langle \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), -1\rangle$.
- For $k = \mathbb{F}_q$, to compute $\#X_H(k) = \#Y_H(k) + \#X_H^\infty(k)$ count double cosets fixed by G_k.
- See Drew's Slides for a nice summary of the implementation.

Arithmetically maximal H of ℓ-power level for which $X_H(\mathbb{F}_p) = \emptyset$ for some $p \neq \ell \leq 37$

label	level	generators	p	rank	genus
16.48.2.17	2^4	$\begin{pmatrix} 11 & 9 \\ 4 & 13 \end{pmatrix}, \begin{pmatrix} 13 & 5 \\ 4 & 11 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 12 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 5 \end{pmatrix}$	$3, 11$	0	2
27.108.4.5	3^3	$\begin{pmatrix} 4 & 25 \\ 6 & 14 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 3 & 1 \end{pmatrix}$	$7, 31$	0	4
25.150.4.2	5^2	$\begin{pmatrix} 7 & 20 \\ 20 & 7 \end{pmatrix}, \begin{pmatrix} 22 & 2 \\ 13 & 22 \end{pmatrix}$	2	0	4
25.150.4.7	5^2	$\begin{pmatrix} 24 & 24 \\ 0 & 18 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 0 & 23 \end{pmatrix}$	$3, 23$	4	4
25.150.4.8	5^2	$\begin{pmatrix} 8 & 4 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 16 & 7 \\ 0 & 8 \end{pmatrix}$	2	0	4
25.150.4.9	5^2	$\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 18 \\ 0 & 14 \end{pmatrix}$	2	0	4
49.168.12.1	7^2	$\begin{pmatrix} 39 & 6 \\ 36 & 24 \end{pmatrix}, \begin{pmatrix} 11 & 9 \\ 24 & 2 \end{pmatrix}$	2	3	12
13.84.2.2	13	$\begin{pmatrix} 3 & 7 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 12 & 4 \\ 0 & 12 \end{pmatrix}$	2	0	2
13.84.2.3	13	$\begin{pmatrix} 9 & 2 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 0 & 7 \end{pmatrix}$	3	0	2
13.84.2.4	13	$\begin{pmatrix} 8 & 12 \\ 0 & 10 \end{pmatrix}, \begin{pmatrix} 8 & 3 \\ 0 & 9 \end{pmatrix}$	2	0	2
13.84.2.6	13	$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 11 & 3 \\ 0 & 10 \end{pmatrix}$	3	0	2

Decomposing the Jacobian of X_H

Let H be an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ of level N.
Let J_H denote the Jacobian of X_H.

Theorem (Rouse–Sutherland–Voight–ZB 2021)

*Each **simple factor** A of J_H is **isogenous** to A_f for a weight-2 eigenform f on $\Gamma_0(N^2) \cap \Gamma_1(N)$.*

Corollary (Kolyvagin's theorem)

*If A is an **isogeny factor** of X_H, and if the **analytic rank** of A is **zero**, then $A(\mathbb{Q})$ is finite.*

Corollary (Decomposition)

*We can **decompose** J_H up to isogeny using linear algebra and point-counting.*

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety.**

$$
\begin{array}{c}
X(\mathbb{Q}) \\
\big\downarrow \\
X(\mathbb{F}_p)
\end{array}
$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty.**

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety**.

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d] & A(\mathbb{Q}) \arrow[d, "\beta"] \\ X(\mathbb{F}_p) \arrow[r, "\pi"] & A(\mathbb{F}_p). \end{tikzcd}$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty.**
- If $\mathrm{im}\,\pi \cap \mathrm{im}\,\beta$ is **empty** then $X(\mathbb{Q})$ is **empty.**

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety**.

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d] & A(\mathbb{Q}) \arrow[d, "\beta"] \\ \prod_{p\in S} X(\mathbb{F}_p) \arrow[r, "\pi_S"] & \prod_{p\in S} A(\mathbb{F}_p). \end{tikzcd}$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty.**
- If $\operatorname{im}\pi \cap \operatorname{im}\beta$ is **empty** then $X(\mathbb{Q})$ is **empty**.
- This is explicit and is implemented in Magma.

An equationless sieve for the group `121.605.41.1`

The curve X_H has **local points everywhere**, and analytic **rank = genus = 41**.

$H(11) \subset N_{\mathrm{ns}}(11)$, so X_H maps to $X_{\mathrm{ns}}^+(11)$, which is an elliptic curve of rank 1.

$$\begin{tikzcd} X_H(\mathbb{Q}) \arrow[r] \arrow[d] & X_{\mathrm{ns}}^+(11)(\mathbb{Q}) \arrow[r, equal] \arrow[d, "\beta"] & \langle R\rangle \cong \mathbb{Z} \\ \prod_{p\in S} X_H(\mathbb{F}_p) \arrow[r, "\pi_S"] & \prod_{p\in S} X_{\mathrm{ns}}^+(11)(\mathbb{F}_p) & \end{tikzcd}$$

We can compute $\operatorname{im}\pi_S$ without *equations* for X_H or π_S

- A point of $X_{\mathrm{ns}}^+(11)(\mathbb{F}_p)$ corresponds to E with $\rho_{E,11}(G_{\mathbb{F}_p}) \subset N_{\mathrm{ns}}(11)$ and lifts to a point of $X_H(\mathbb{F}_p)$ if and only if $\rho_{E,121}(G_{\mathbb{F}_p}) \subset H(121)$.

- For $p = 13$ the image of any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 1, 5 \bmod 7$.
- For $p = 307$ any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 2, 3, 4, 7, 10, 13 \bmod 14$.
- Therefore $Y_H(\mathbb{Q}) = \emptyset$ (and in fact $X_H(\mathbb{Q}) = \emptyset$; there are no rational cusps).

Gargantuan models of modular curves[1]

- We computed canonical models (over $\mathbb{Q}$) for `27.729.43.1` (resp. `25.625.36.1`).

- We use these models to prove that X_H has no $\mathbb{Q}_3$ (resp. $\mathbb{Q}_5$) as follows.

- These models have very bad reduction at $p = 3$ (resp. 5). (They're not even flat.)

- $X_H(\mathbb{F}_p) \neq \emptyset$ for all p, but $X_H(\mathbb{Z}/p^2\mathbb{Z}) = \emptyset$ for $p = 3$ (resp. 5).

- The "Greenberg transform" (i.e., the "Wittferential tangent space" of Buium) is adjoint to Witt vectors: $X_H^{(1)}(\mathbb{F}_p) = X_H(\mathbb{Z}/p^2\mathbb{Z})$.

- The fibers of the map $X_H^{(1)} \to X_H$ have no $\mathbb{F}_p$ points.

[1] We give thanks to Poonen and Zywina

Subgroups of $\mathrm{GL}_2(\mathbb{Z}_2)$