Diophantine Geometry and Uniformity

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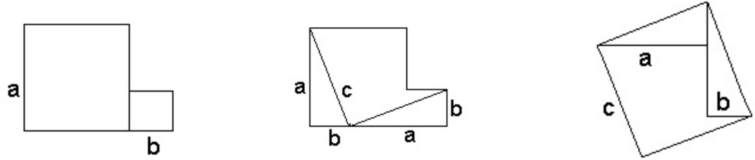
Slides available at http://dmzb.github.io/

Euler Circle Colloquium
June 19, 2025

$$\boxed{a^2 + b^2 = c^2}$$



Basic Problem (Solving Diophantine Equations)

Let $f_1, \ldots, f_m$ be polynomials with integer coefficients, e.g.,

$$x^2 + y^2 + 1$$
$$x^3 - y^2 - 2$$
$$2y^2 + 17x^4 - 1$$

Basic problem: solve polynomial equations

Describe the set

$$V(f_1, \ldots, f_m) = \left\{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \forall i,\, f_i(a_1, \ldots, a_n) = 0\right\},$$

i.e., the set of integer solutions to those polynomials

Fact

Solving Diophantine equations is difficult.

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Hilbert's Tenth Problem

Theorem (Davis–Putnam–Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

***input**: integer polynomials $f_1, \ldots, f_m$ in variables $x_1, ..., x_n$;*

***output**: $\mathrm{YES} / \mathrm{NO}$ according to whether the set of solutions*

$$\{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \ldots, a_n) = 0\}$$

is non-empty.

This is *known* to be true for many other cases (e.g., $\mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).

This is *still unknown* in many other cases (e.g., $\mathbb{Q}$).

Fermat's Last Theorem - A Marvelous Proof

Theorem (Wiles; Taylor)

For primes $p \geq 3$ the only integer solutions to the equation

$$x^p + y^p = z^p$$

are integer multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad \pm(0,1,1).$$

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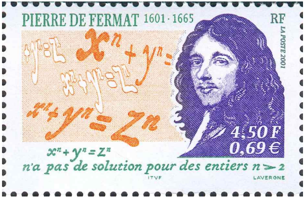

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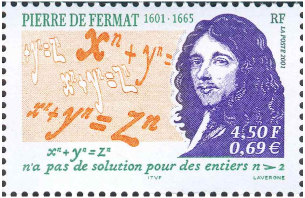

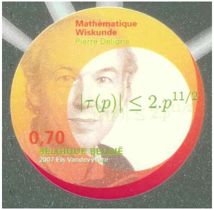

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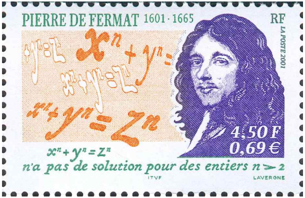
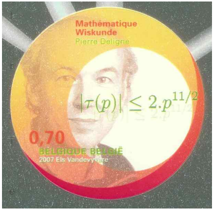
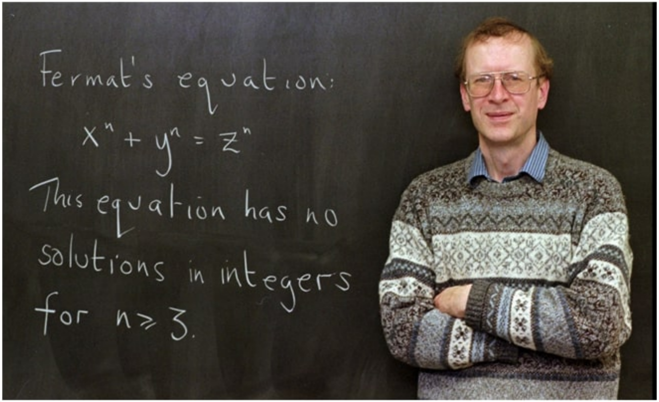




https://mathshistory.st-andrews.ac.uk/Miller/stamps/

Fermat's Last Theorem - aftermath



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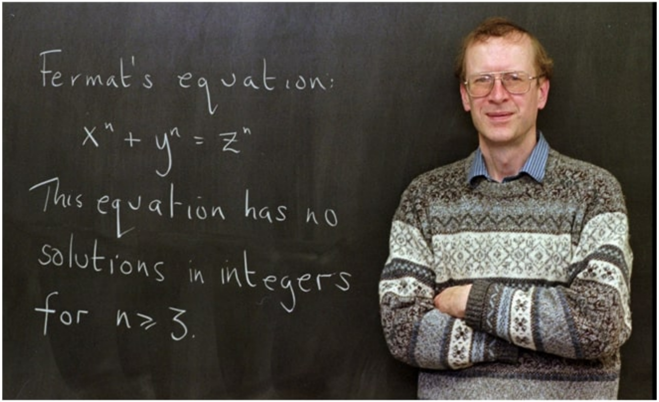

GAP

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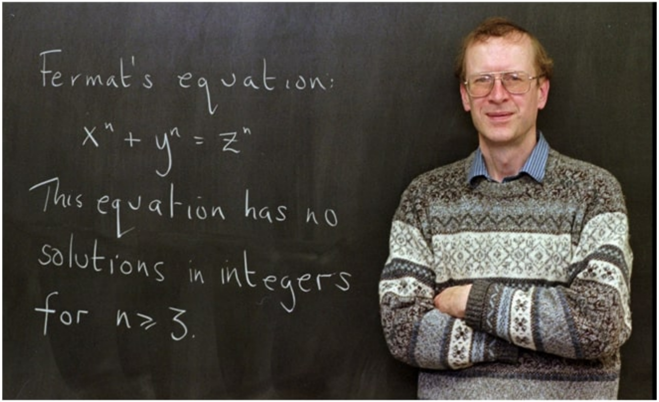
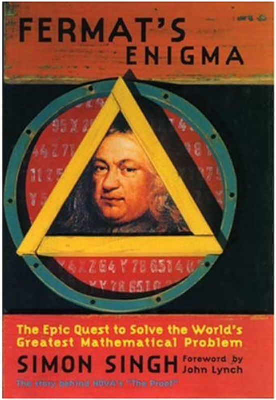
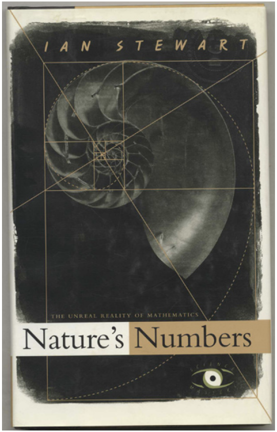
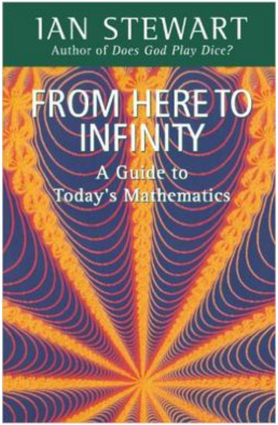


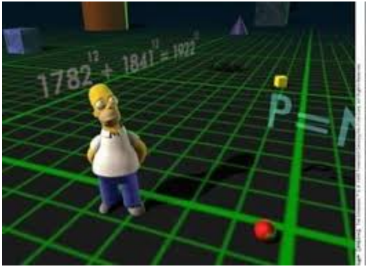
Books


FERMAT'S
ENIGMA
The Epic Quest to Solve the World's Greatest Mathematical Problem
SIMON SINGH
Foreword by John Lynch
The story behind NOVA's "The Proof"


IAN STEWART
THE UNREAL REALITY OF MATHEMATICS
Nature's Numbers


IAN STEWART
Author of Does God Play Dice?
FROM HERE TO INFINITY
A Guide to Today's Mathematics

Fermat trolling



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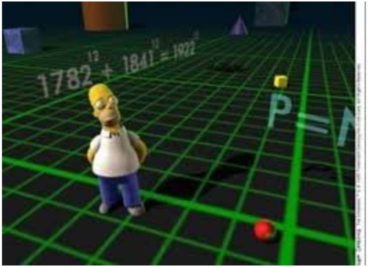

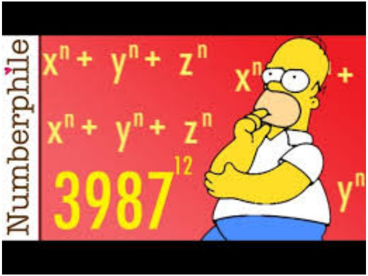

See https://youtu.be/ReOQ300AcSU?si=--fAdsdPttt4HR3N

Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

Qualitative:

- Does there exist a solution?
- Do there exist infinitely many solutions?
- Does the set of solutions have some extra structure (e.g., geometric structure, group structure).

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- How large is the smallest solution?
- How can we explicitly find all solutions? (With proof?)

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Implicit question

- Why do equations have (or fail to have) solutions?
- Why do some have many and some have none?
- What underlying mathematical structures control this?

Example: Pythagorean triples

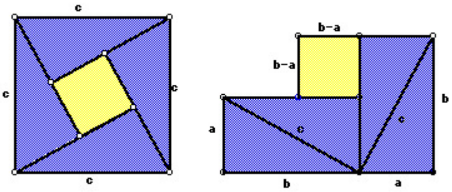
$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$7^2 + 24^2 = 25^2$$

Lemma

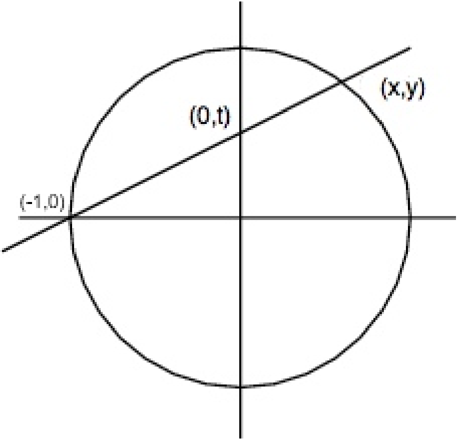
The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.



Pythagorean triples



Slope = $t = \frac{y}{x+1}$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

(with $c \neq 0$) are all multiples of the triples

$$\boxed{a = 1 - t^2} \ \boxed{b = 2t} \ \boxed{c = 1 + t^2}$$

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

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Theorem (Faltings)

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Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4.

Fermat Curves

Question

Why is Fermat's last theorem believable?

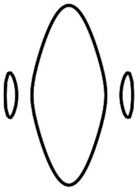
1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

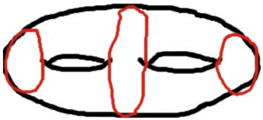
Mordell Conjecture

Example

$$y^2 = -(x^2 - 1)(x^2 - 2)(x^2 - 3)$$

This is a cross section of a two holed torus.





The **genus** is the number of holes.

Conjecture (Mordell, 1922)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

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2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)
3. and has genus

$$(n-1)(n-2)/2$$

which is ≥ 2 iff $n \geq 4$.

Fermat Curves

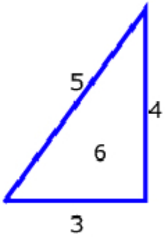
Question

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

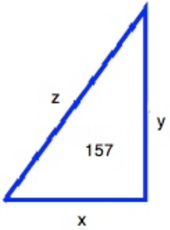
$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 6}$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$



Assume the Birch–Swinnerton-Dyer conjectures

If you assume $1,000,000 worth of conjectures, then the equations

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"Next" soluton has **176 digits**!

Back of the envelope calculation (as of 2011)

$$\boxed{x^2 + y^2 = z^2,\, xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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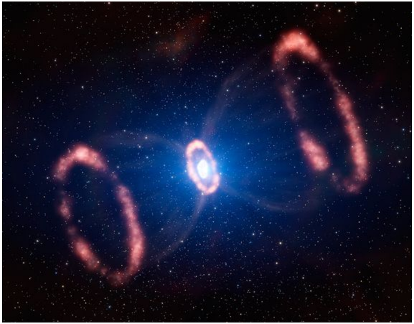
- Expected time until 'heat death' of universe – $\mathbf{10^{100}}$ **years.**



Fermat Surfaces

Conjecture

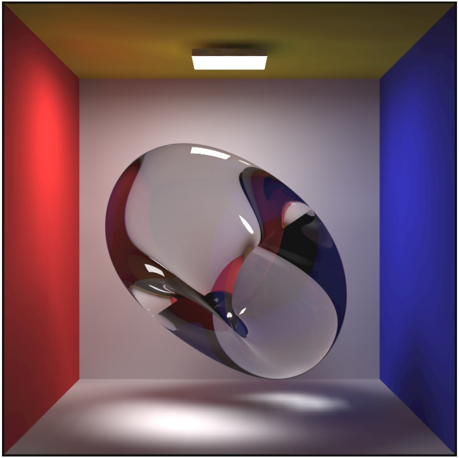
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ *or lie on the lines 'lines'* $x = z$, $y = w$ *(and permutations).*

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



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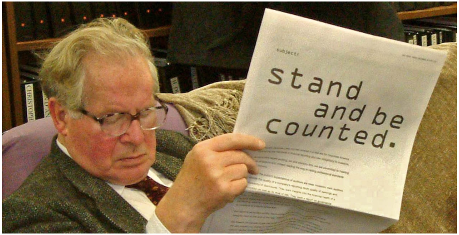

Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$

- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $\left(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407}\right)$.

Problem

Find another solution. (Probably impossible.)

Back of envelope calcluation

1. $\mathbf{10^{16}}$ **years** to find via brute force.
2. Age of the universe – $\mathbf{13.75 \pm .11}$ **billion years** (roughly $\mathbf{10^{10}}$).

Sums of cubes

$$1 = 1^3 + 0^3 + 0^3$$
$$2 = 1^3 + 1^3 + 0^3$$
$$3 = 1^3 + 1^3 + 1^3$$
$$3 = 4^3 + 4^3 + (-5)^3$$
$$4 \neq x^3 + y^3 + z^3$$
$$5 \neq x^3 + y^3 + z^3$$
$$6 = 1^3 + 1^3 + 2^3$$

Conjecture (Heath-Brown)

The equation

$$x^3 + y^3 + z^3 = n$$

has an integer solution if and only if n is not 4 or 5 mod 9.

Solved by Booker–Sutherland

$32 \neq x^3 + y^3 + z^3$

$33 =$

https://www.quantamagazine.org/why-the-sum-of-three-cubes-is-a-hard-math-problem-20191105/

Solved by Booker–Sutherland

$$32 \neq x^3 + y^3 + z^3$$

$$33 = 8866128975287528^3 + (-8778405442862239)^3 + (-2736111468807040)^3$$

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“Generalized” Fermat equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

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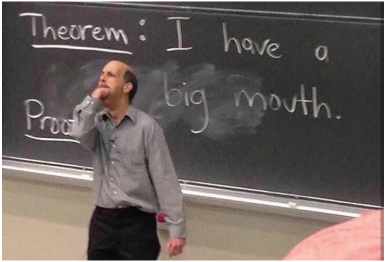

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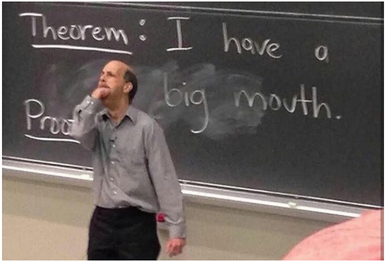

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Problem

What are the solutions to the equation $x^a + y^b = z^c$?

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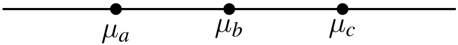
Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$*. Then the equation* $x^a + y^b = z^c$ *has only finitely many coprime integer solutions iff* $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$*.*



Known Solutions to $x^a + y^b = z^c$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

$$1^p + 2^3 = 3^2, \qquad 2^5 + 7^2 = 3^4$$

$$7^3 + 13^2 = 2^9, \qquad 2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

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Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman–Zagier)

This is a complete list of coprime non-zero solutions such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

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Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < \frac{2}{4} - \frac{1}{2} = 0$$

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The only Fibonacci numbers that are perfect powers are

$$F_1 = F_2 = 1, \; F_6 = 8, \; F_{12} = 144.$$

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

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Theorem (Silliman–Vogt; 2013 REU)

0 and 1 are the only perfect powers in the Lucas sequence

$$L_1 = 0, L_2 = 1, \quad L_n = 3L_{n-1} - 2L_{n-2}.$$

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, \ldots, 2^n - 1, \ldots$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of f,
3. $T =$ a degree 3 covariant of the dodecahedron.

(a,b,c) such that $\chi < 0$ and the solutions to $x^a + y^b = z^c$ have been determined.

$\{n,n,n\}$	Wiles,Taylor–Wiles, building on work of many others
$\{2,n,n\}$	Darmon–Merel, others for small n
$\{3,n,n\}$	Darmon–Merel, others for small n
$\{5,2n,2n\}$	Bennett
$(2,4,n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2,n,4)$	Bennett–Skinner; $n \geq 4$
$\{2,3,n\}$	Poonen–Shaefer–Stoll, Bruin. $6 \leq n \leq 9$
$\{2,2\ell,3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
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$(2,3,10)$	**ZB**

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve with genus at least 2. Then $\#X(\mathbb{Q}) < \infty$.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Conjecture (Lang, Vojta)

Let X be a variety of general type. Then $X(\mathbb{Q})$ is not (Zariski) dense.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g+1)$

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Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a square.

Main Theorem (uniformity for curves of small rank)

Theorem (Katz–Rabinoff–ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

Chabauty's method

(**p-adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}),\ \omega \in V.$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#\left(\operatorname{div}\omega \cap D_P\right)$

(**Riemann–Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

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Example

$$X \colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

1. Points reducing mod 3 to $\widetilde{Q} = (0, 1)$ are given by

$$\begin{aligned} x &= 3 \cdot t, \text{ where } t \in \mathbb{Z}_3 \\ y &= \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

2. $$\int_{(0,1)}^{P_t} \frac{x dx}{y} = \int_0^t (x - x^3 + \cdots) dx$$

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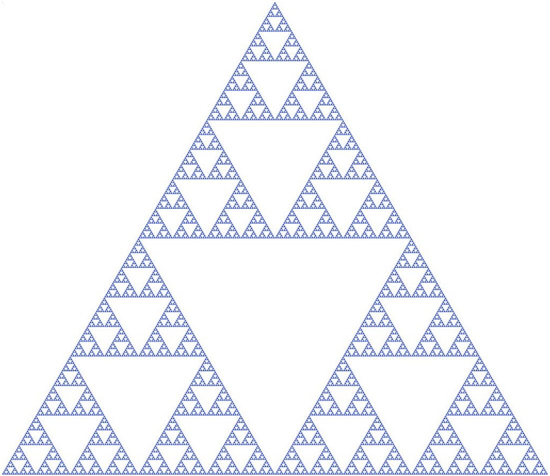
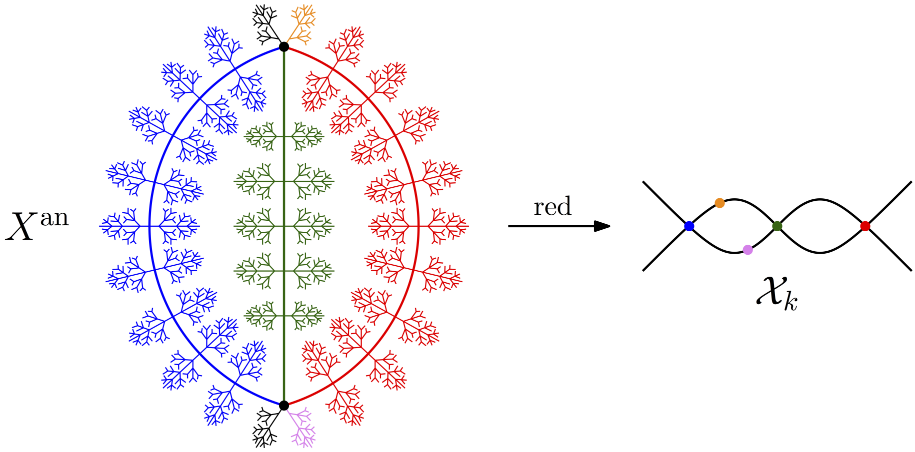
3-adics vs Sierpinski triangle

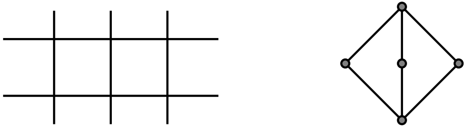


Berkovich picture



Baker–Payne–Rabinoff and the slope formula

(**Dual graph** Γ **of** $X_{\mathbb{F}_p}$)



(**Contraction Theorem**) $\tau \colon X^{\mathrm{an}} \to \Gamma$.

(**Combinatorial harmonic analysis/potential theory**)

f	a meromorphic function on X^{an}
$F := (-\log\lvert f \rvert)\big\rvert_{\Gamma}$	associated tropical, piecewise linear function
$\operatorname{div} F$	combinatorial record of the slopes of F

(**Slope formula**) $\tau_* \operatorname{div} f = \operatorname{div} F$

Berkovich picture