The canonical ring of a stacky curve

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Slides available at `https://dmzb.github.io/`

Modular forms

Let Γ be a Fuchsian group (e.g. $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$).

Definition

A **modular form** for Γ of weight $k \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ such that

$$f(\gamma z) = (cz + d)^k f(z) \quad \text{for all } \gamma \in \Gamma$$

and such that the limit $\lim_{z \to *} f(z)$ exists for all cusps $*$.

Definition

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms for Γ of weight k.

Ring of Modular forms

Definition (Ring of Modular forms)

$$M(\Gamma) := \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma)$$

Example

$$M(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6]$$

Theorem (Wagreich)

$M(\Gamma)$ is generated by two elements if and only if

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}), \Gamma_0(2), \textit{or } \Gamma(2).$$

Ring of Modular forms

Definition (Ring of Modular forms)

$$M(\Gamma) := \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma)$$

Example (LMFDB)

$$M(\Gamma_0(11)) \cong \mathbb{C}[E_2, f_E, g_4]/(g_4^2 - F(E_2, f_E))$$

Example (Ji, 1998)

$$M(\Gamma_{2,3,7}) \cong \mathbb{C}[\Delta_{12}, \Delta_{16}, \Delta_{30}]/f(\Delta_{12}, \Delta_{16}, \Delta_{30})$$

Rustom's conjectures (2012)

Conjecture (Rustom)

The $\mathbb{C}$-algebra $M(\Gamma_0(N))$ is generated in weight at most 6 with relations in weight at most 12.

– This was proved by Wagreich in 1980/81.

Conjecture (Rustom)

The $\mathbb{Z}[1/6N]$-algebra $M(\Gamma_0(N), \mathbb{Z}[1/6N])$ is generated in weight at most 6 with relations in weight at most 12.

– $M_k(\Gamma_0(N), R)$ consists of forms with q-expansion in $R[\![q]\!]$.

Main Theorem

Conjecture (Rustom)

The $\mathbb{Z}[1/6N]$-algebra $M(\Gamma_0(N), \mathbb{Z}[1/6N])$ is generated in weight at most 6 with relations in weight at most 12.

Theorem (Voight, ZB)

Rustom's conjecture is true.

Theorem (Voight, ZB)

More generally, the $\mathbb{C}$-algebra $M(\Gamma, \mathbb{C})$ is generated in weight at most $6e$ with relations in weight at most $12e$, where e is the max of the orders of the stabilizers of Γ.

Translation to Geometry (Kodaira–Spencer)

Modular curves

1. $Y = [\mathcal{H}/\Gamma]$
2. $\Delta =$ cusps
3. $X = Y \cup \Delta = [\overline{\mathcal{H}}/\Gamma]$

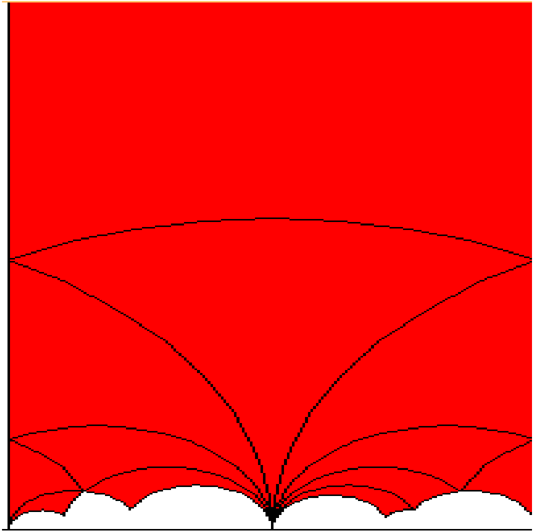
Kodaira–Spencer

$$M_k(\Gamma) \cong H^0(X, \Omega^1(\Delta)^{\otimes k/2})$$
$$f(z) \mapsto f(z)\, dz^{\otimes k/2}$$

Log canonical ring

$$M(\Gamma) \cong R_{X,\Delta} := \bigoplus_k H^0(X, \Omega^1(\Delta)^{\otimes k})$$

Example: $X_0(11)$ (fundamental domain)



Example: $X_0(11)$, $\Delta = P + Q$

Example (LMFDB)

$$\bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_0(11)) \cong \mathbb{C}[E_2, f_E, g_4]/(g_4^2 - F(E_2, f_E))$$

Remark (Via Kodaira Spencer)

$$\bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_0(11)) \cong \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X_0(11), k(P+Q))$$

Remark (Riemann–Roch)

$$\dim H^0(X_0(11), k(P+Q)) = \max\{1, 2k\}$$

$$\dim \operatorname{im}\left(H^0(X_0(11), P+Q)^{\otimes^2} \to H^0(X_0(11), 2(P+Q))\right) = 3$$

Log canonical map/ring

Definition

The **canonical map** $\phi_K \colon C \to \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \ldots : \omega_g(P)]$.

(An embedding iff C is not hyperelliptic.)

Facts

$$C \cong \operatorname{Proj} R_{X,\Delta} \cong \operatorname{Proj} \bigoplus_k H^0(X, \Omega^1(\Delta)^{\otimes k})$$

Facts

The relations among $R_{X,1}$ are the defining equations of $\phi_K(C)$.

Petri's theorem

Let C be non-hyperelliptic, non-trigonal, not a plane quintic.

Theorem (Enriques-Noether-Baggage-Petri)

The canonical ring R_C is generated in degree 1 with relations in degree 2.

Remark

1. For C trigonal or a plane quintic R_C is generated in degree 1 with relations in degrees 2 and 3
2. (unless $g(C) = 3$, which has a single relation in degree 4)
3. For C hyperelliptic, there are generators in degrees 1,2, relations in degrees up to 4.

Log Petri's theorem

Let C be a curve and Δ a log divisor.

Theorem (Voight, ZB)

The log canonical ring R_C is generated in degree at most 3 with relations in degree at most 6.

Remark

Lots of exceptional cases if $0 < \deg \Delta \leq 2$.

Remark (Things stabilize)

1. Generators in degree 1 with relations in degree 2,3 if $\Delta = 3$
2. (Mumford.) Generators in degree 1 with relations in degree 2 if $\Delta \geq 4$

Log Petri's theorem

Let C be a curve and Δ a log divisor.

Theorem (Voight, ZB)

The log canonical ring R_C is generated in degree at most 3 with relations in degree at most 6.

Corollary

Rustom's conjecture is true if Γ acts without stabilizers.

Translation to Geometry (Kodaira–Spencer)

Modular curves

1. $Y = [\mathcal{H}/\Gamma]$
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3. $X = Y \cup \Delta = [\overline{\mathcal{H}}/\Gamma]$

Kodaira–Spencer

$$M_k(\Gamma) \cong H^0(X, \Omega^1(\Delta)^{\otimes k/2})$$
$$f(z) \mapsto f(z)\, dz^{\otimes k/2}$$

Log canonical ring

$$M(\Gamma) \cong R_{X,\Delta} := \bigoplus_k H^0(X, \Omega^1(\Delta)^{\otimes k})$$

Fundamental Domain for $X(1)$

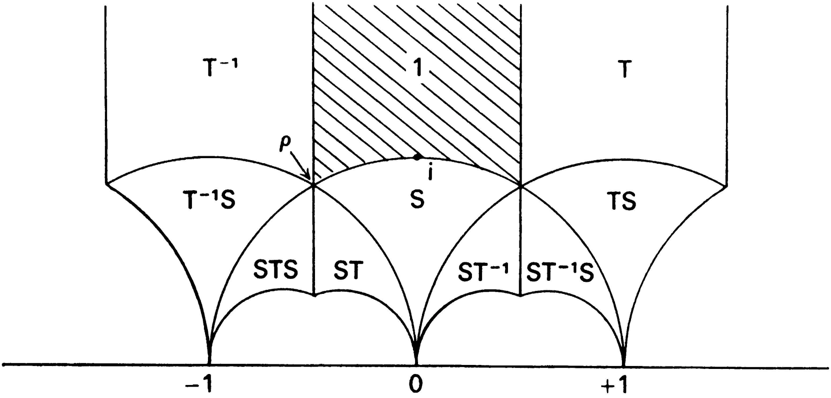


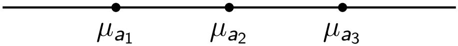
Fig. 1

Fundamental Domain for $X(1)$

$$D = K + \Delta = -\infty$$

d	dD	$\dim H^0(X, \lfloor dD \rfloor)$	$\dim M_{2d}(\mathrm{SL}_2(\mathbb{Z}))$
0	0	1	1
1	$-\infty$	0	0
2	-2∞	0	1
3	-3∞	0	1
4	-4∞	0	1
5	-5∞	0	1
6	-6∞	0	2

Fractional divisors



Remark

1. Divisors are now **fractional**.
2. $D = D_0 + \frac{n_1}{a_1}P_1 + \frac{n_2}{a_2}P_2 + \frac{n_3}{a_3}P_3$

Fact

$$K_{\mathscr{X}} = K_X + \sum \frac{e_P - 1}{e_P} P$$

Floors

Definition

The **floor** $\lfloor D \rfloor$ of a Weil divisor $D = \sum_i a_i P_i$ on $\mathscr{X}$ is the divisor on X given by

$$\lfloor D \rfloor = \sum_i \left\lfloor \frac{a_i}{\# G_{P_i}} \right\rfloor \pi(P_i).$$

Fact

$$H^0(\mathscr{X}, D) = H^0(X, \lfloor D \rfloor)$$

Example: $X(1)$

$$D = K + \Delta = \tfrac{1}{2}P + \tfrac{2}{3}Q - \infty$$

d	$\lfloor dD \rfloor$	$\deg \lfloor dD \rfloor$	$\dim H^0(X, \lfloor dD \rfloor)$	$M_{2d}(\mathrm{SL}_2(\mathbb{Z}))$
0	0	0	1	1
1	$-\infty$	-1	0	0
2	$P + Q - 2\infty$	0	1	E_4
3	$P + 2Q - 3\infty$	0	1	E_6
4	$2P + 2Q - 4\infty$	0	1	E_4^2
5	$2P + 3Q - 5\infty$	0	1	$E_4 E_6$
6	$3P + 4Q - 6\infty$	1	2	E_4^3, E_6^2

Main theorem

Theorem (Voight,ZB)

Let $(\mathscr{X}, \Delta)$ *be a tame log stacky curve with signature* $(g; e_1, \ldots, e_r; \delta)$ *over a field* k*, and let* $e = \max(1, e_1, \ldots, e_r)$*. Then the canonical ring*

$$R(\mathscr{X}, \Delta) = \bigoplus_{d=0}^{\infty} H^0(\mathscr{X}, \Omega(\Delta)^{\otimes d})$$

is generated as a k*-algebra by elements of degree at most* $3e$ *with relations of degree at most* $6e$*.*

Remark

Moreover, if $2g - 2 + \delta \geq 0$, then $R(\mathscr{X}, \Delta)$ is generated in degree at most $\max(3, e)$ with relations in degree at most $2\max(3, e)$.

Final comments

Remark

1. We generalize to the relative and spin cases.
2. We give (relative) Gröbner bases, generic initial ideals.
3. Exact formulations of theorems are amenable to computation.