MATH 220 HANDOUT 3 - PROOF BY CONTRADICTION

(1) Prove that if $x + y > 5$, then $x > 2$ or $y > 3$.
(2) Let $0 < \alpha <1$. Prove that $\sqrt{\alpha} > \alpha$.
(3) Prove that there are no integer solutions to the equation $x^2 = 4y + 2$
(4) Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
(5) Prove that there is no smallest positive real number.
(6) Let b_1, b_2, b_3, b_4 be positive integers such that

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} = 1.$$

Prove that at least one of the b_k's is even. Hint: clear the denominators.
(7) Show that if a is rational and b is irrational, then $a + b$ is irrational.
(8) Prove that $\sqrt{3}$ is irrational.
(9) Prove that if $r^3 + r + 1 = 0$ then r is irrational.
(10) Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even. (Harder problem: show that a or b must be even.)
(11) If a, b, c are odd integers, prove that $ax^2 + bx + c = 0$ does not have a solution x such that x is a rational number.
(12) Prove that if $3 \mid (a^2 + b^2)$, then $3 \mid a$ and $3 \mid b$. Hint: If $3 \nmid a$ and $3 \nmid b$, what are the possible remainders of a, b, a^2, and b^2 upon division by 3?
(13) Prove that $\log_{10} 7$ is irrational.
(14) Let $b \in \mathbf{Z}_{\geq 1}$. Prove that $\log_b 3/\log_b 2$ is irrational.
(15) Prove that $\sqrt[5]{5}$ is irrational.
(16) Prove that the equation

$$(x^2 - y^2)(x^2 - 4y^2) = 7$$

has no solutions with $x, y \in \mathbf{Z}$.
(17) Prove that there are infinitely many primes of the form $6n + 1$ or there are infinitely many primes of the form $6n + 5$.
(18) Prove that there are infinitely many primes of the form $6n + 5$.
(19) Try to prove that there are infinitely many primes of the form $6n + 1$. What goes wrong in the argument from the previous problem?
(20) Prove that if n ≥ 2, then $\sqrt[n]{n}$ is irrational. Hint: use that if $n > 2$, then $2^n > n$.