**MATH 250, Foundations of Mathematics**
**TuTh 2:30 - 3:45**

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**All assignments**
Last updated: December 5, 2022

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# Contents

# Assignment 1

**Topics**: Introduction to the course. Mathematical reasoning.

**Reading**: Chapter 1, except for proof by contradiction.

**Suggested problems (do not hand in)**

- With answers:
  - Section 1.1, #1(adgj), 2(adji), 3(adgi), 5(ad), 6(a)
  - Section 1.2, #2(ac), 4(ac), 5(ad), 7(a), 10(a), 11(a), 12(a)
  - Section 1.3, #1(ad), 3(a), 5(ac), 7(ac)
  - Section 1.4, #1, 4(a), 6(a), 8, 12(ab), 15(a)
- Without answers: Handout 1

**Assignment, due September 6, via Canvas**:

1. Suppose that $n$ is an even integer, and let $m$ be any integer. Prove that $nm$ is even.
2. Suppose that $n$ is an odd integer. Prove that $n^2$ is an odd integer. (Hint: an integer $n$ is odd if and only if there exists an integer $k$ such that $n = 2k + 1$.)
3. Prove that if $n^2$ is even, then $n$ is even. (Hint: see Section 1.4)
4. Write the negation of each of the following statements.
   (a) All triangles are isosceles.
   (b) Every door in the building was locked.
   (c) Some even numbers are multiples of three.
   (d) Every real number is less than 100.
   (e) Every integer is positive or negative.
   (f) If $f$ is a polynomial function, then $f$ is continuous at 0.
   (g) If $x^2 > 0$, then $x > 0$.
   (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.
   (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$
   (j) $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.

# Assignment 2

**Topics**: “Basic” proofs and divisibility problems.

**Reading**:

- Finish reading chapter 1.
- Section 5.3

**Suggested problems (do not hand in)**

1. With answers: Section 5.3, #1(a), 4(a), 6(ac)
2. Without answers: Section 5.3, #2, 4 (without induction), 5 (without induction)
3. Handout 2

**Assignment, due Tuesday, September 13, via Canvas**:

1. Prove that if $x$ is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: $x$ is even, $x$ is odd. Also, feel free to use basic facts about even or odd, e.g., “odd + odd = even”, without additional proof.)
2. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
3. Show that for all integers $a$ and $b$,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
4. Find all positive integers $n$ such that $n^2 - 1$ is prime. Prove that your answer is correct.

# Assignment 3

**Topics**: Proof by contradiction. Unsolvability of equations. Irrationality.

**Reading**:

- Section 1.4, p. 41-42 (stop at Historical Comments)
- Section 5.4

**Suggested problems (do not hand in)**

1. Without answers: Section 1.4 #21
2. Without answers: Section 5.4 #6, 7, 10(a), 15, 18,
3. Handout 3

**Assignment, due Tuesday, September 20, via Canvas**:

1. Prove that $2^{1/3}$ is irrational.
2. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
3. Let $a, b, c$ be integers satisfying $a^2 + b^2 = c^2$. Show that $abc$ must be even. (Harder problem, just for fun: show that $a$ or $b$ must be even.)
4. Suppose that $a$ and $n$ are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and $n$ is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)

# Assignment 4

**Topics**: Induction.

**Reading**: Section 5.2, p. 159-163

**Fun Video**: Vi Hart; "Doodling in Math: Spirals, Fibonacci, and Being a Plant"
https://www.youtube.com/watch?v=ahXIMUkSXX0

**Suggested problems (do not hand in)**

1. With answers: Section 5.2 #1(a), 4(a), 8(ad), 9(a), 29
2. Without answers: Section 5.2 #2-9, 13
3. Handout 4
4. Handout 5

**Assignment, due Tuesday, September 27, via Canvas**:

1. Prove that for every positive integer $n$,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let $a_n$ be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers $n$, $a_n < 2$.
3. Prove by induction that if $b_1, b_2, \ldots, b_n$ are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.
4. Let $F_1, F_2, F_3, \ldots = 1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

# Assignment 5

**Topics**: Basics of set theory. Basic operations. Proofs with sets.

**Reading**:

1. Section 2.1, p. 49-57;
2. Section 2.2, p. 61-65 (stop at DeMorgan's laws)

**Suggested problems (do not hand in)**

1. With answers (many of these are calculations; do as many as you need to do to understand the definitions):
   (a) Section 2.1, #1(adg), 2(adg), 4(adg), 5(a), 7(a), 8(ae), 9(adf), 10(a), 18(acf), 19(ad), 20(ae), 21
   (b) Section 2.2, #1(adgj), 2(ad), 4(ad), 5(ad), 7(a), 9(ad), 14(a),
2. Without answers:
   (a) Section 2.1, 13, 14, 15, 16,
   (b) Section 2.2, #1-12
3. Handout 6

**Assignment, due Tuesday, October 4, via Canvas**:

1. Let $A = \{n \in \mathbb{Z} | n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} | n^2 \text{ is a multiple of } 4\}$
   (a) Prove or disprove: $A \subseteq B$.
   (b) Prove or disprove: $B \subseteq A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let $A, B$ and $C$ be sets.
   (a) Prove that $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.
   (b) State the contrapositive of part (a).
   (c) State the converse of part (a). Prove or disprove it.
4. Let $n$ and $m$ be integers. Prove that if $n\mathbb{Z} \subseteq m\mathbb{Z}$ then $m$ divides $n$.

# Assignment 6

**Fall Break is Monday October 10 and Tuesday, October 11; no class or office hours these days**.
Office hours will be Wednesday, October 12 (4:30-5:30 via Zoom), and the assignment will be due Thursday October 13.

**Topics**: More proofs with sets. DeMorgan's laws. Cartesian Products. Power sets

**Reading**:

1. Section 2.2, p. 65-66;
2. Section 2.3, p. 72, just the part about power sets.

**Suggested problems (do not hand in)**

1. With answers:
   (a) Section 2.2, 13(a), 16(a)
   (b) Section 2.3, #1(a), 3, 5(adg),
2. Without answers:
   (a) Section 2.2, 14, 16-19, 21, 23-27
   (b) Section 2.3, #1(b), 2,4
3. Handout 7

**Assignment, due Thursday, October 13, via Canvas**:

1. Let $A$ and $B$ be sets. Prove that $(A \cup B) \cap \overline{A} = B - A$.
2. Let $A$ and $B$ be sets. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
3. Let $A = \{0, 1, 2\}$. Which of the following statements are true? (No justification is needed.)
   (a) $\{0\} \subseteq P(A)$;
   (b) $\{1, 2\} \in P(A)$;
   (c) $\{1, \{1\}\} \subseteq P(A)$.
   (d) $\{\{0, 1\}, \{1\}\} \subseteq P(A)$;
   (e) $\emptyset \in P(A)$;
   (f) $\emptyset \subseteq P(A)$;
   (g) $\{\emptyset\} \in P(A)$.

(h) $\{\emptyset\} \subseteq P(A)$;

4. Let $A$ and $B$ be sets. Prove that if $A \subseteq B$, then $P(A) \subseteq P(B)$. State the converse of this and prove or disprove it.

# Midterm Review/Midterm

**Topics**: Tuesday, October 18 will be an in class Exam review. We will not cover any new material; in class, I will answer whatever questions you have. **Please show up with questions**.

There will be no office hours on Monday, October 17; instead there will be office hours Wednesday, October 18, 2:30-3:30. The exam is on Thursday, October 20.

**Content**: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

A typical exam will have one or two questions from each week of the course. You can expect problems like the following:

- Negations
- Give definitions
- Contrapositive
- Contradiction
- Induction
- Proofs with sets

# Assignment 7

**Topics**: Introduction to functions; images and surjectivity

**Reading**:

1. Section 3.1, p. 81-90 (stop at “Inverse Image”);
2. Section 3.2, p. 97-100 (stop at Injective Functions).

**Suggested problems (do not hand in)**

1. With answers:
   (a) Section 3.1, #1(adg), 4(ace), 5(a), 8(a), 10(a), 12(1d)
   (b) Section 3.2, #1(adgj), 2(ad)
2. Without answers:
   (a) Section 3.1, 1-4,6-13
   (b) Section 3.2, 1-6
   (c) Handout 9

**Assignment, due Tuesday, November 1, via Canvas**:

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 6x + 5$.
   (a) Prove that $f(\mathbf{R}) = \mathbf{R}$.
   (b) Compute $f([1, 4])$. Prove your answer.
2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $x^4 + x^2$.
   (a) Compute the image of $f$. Prove that your answer is correct.
   (b) Compute $f([-1, 2])$. Prove that your answer is correct.
3. Let $A$ and $B$ be sets and let $X$ and $Y$ be subsets of $A$. Let $f\colon A \to B$ be a function. Prove or disprove each of the following. When giving a disproof, please give an counterexample.
   (a) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.
   (b) $f(X \cap Y) \supset f(X) \cap f(Y)$.
   (c) $f(X) - f(Y) \subseteq f(X - Y)$.
   (d) $f(X) - f(Y) \supset f(X - Y)$.

# Assignment 8

**Topics**: Inverse Image (or "Preimage").

**Reading**: Section 3.1, p. 90-92 (stop at the Historical Comments. Or don't.)

**Suggested problems (do not hand in)**

1. With answers: Section 3.1, #17(ad), #18(adg), #19(a), #21(a)
2. Without answers: 17-21
3. Handout 10

Due to a conflict, **office hours** will be 4:00-5pm on Tuesday, November 8 (over Zoom). This is after the homework assignment is due; due to this inconvenience, you are welcome to turn in the assignment late (anytime before Thursday, November 10). (Canvas will mark this as late, but I will not deduct any points.)

**Assignment, due Tuesday, November 8, via Canvas**:

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 3x + 1$.
   (a) Compute $f^{-1}(\{1, 5, 8\})$ (do not give a proof).
   (b) Compute $f^{-1}(W)$, where $W = (4, \infty)$, and give a proof that your answer is correct.
   (c) Compute $f^{-1}(\mathbf{E})$, where $\mathbf{E}$ is the set of even integers, and give a proof that your answer is correct.
2. Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 2n + 4, & \text{if } n \text{ is odd.} \end{cases}$

   Compute $f^{-1}(\mathbf{E})$. Prove that your answer is correct. (Reminder: $\mathbf{E}$ is the set of even integers.)
3. Let $A$ and $B$ be sets and let $X$ be a subset of $B$. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $X \subseteq f(f^{-1}(X))$.
   (b) $X \supset f(f^{-1}(X))$.
4. Let $A$ and $B$ be sets. Let $S \subseteq A$ and let $T \subseteq B$. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $f(S) \subseteq T \Rightarrow S \subseteq f^{-1}(T)$.
   (b) $S \subseteq f^{-1}(T) \Rightarrow f(S) \subseteq T$.

# Assignment 9

**Topics**: Injectivity.

**Reading**: Section 3.2, p. 100-105

**Suggested problems (do not hand in)**

1. With answers: 3.2, #12(adg), #13(bd)
2. Without answers: 3.2 #9-14, 19(abc)
3. Handout 11

**Assignment, due Tuesday, November 15, via Canvas**:

1. Which of the following functions $f\colon \mathbf{R} \to \mathbf{R}$ are injective? If the function is injective, give a proof. If it is not injective, give a counterexample.
   (a) $f(x) = x^4 + x^2$;
   (b) $f(x) = x^3 + x^2$;
   (c) $f(x) = \begin{cases} -x-1, & \text{if } x > 0 \\ x^2, & \text{if } x \leq 0. \end{cases}$
2. Let $A$ and $B$ be sets and let $X$ and $Y$ be subsets of $A$. Let $f\colon A \to B$ be an injective function. Prove that $f(X \cap Y) = f(X) \cap f(Y)$.
3. Let $f\colon A \to B$ be a function. Which of the followings statements are equivalent to the statement '$f$ is injective'? (No proof necessary.)
   (a) $f(a) = f(b)$ if $a = b$;
   (b) $f(a) = f(b)$ and $a = b$ for all $a, b \in A$;
   (c) If $a$ and $b$ are in $A$ and $f(a) = f(b)$, then $a = b$;
   (d) If $a$ and $b$ are in $A$ and $a = b$, then $f(a) = f(b)$;
   (e) If $a$ and $b$ are in $A$ and $f(a) \neq f(b)$, then $a \neq b$;
   (f) If $a$ and $b$ are in $A$ and $a \neq b$, then $f(a) \neq f(b)$.
4. We define a function $f\colon [a, b] \to \mathbf{R}$ to be **decreasing** if for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.
   (a) Negate the definition of decreasing.
   (b) Prove that a decreasing function is injective.

# Assignment 10

**Topics**: Composition of functions.

**Reading**: Section 3.3, p. 110-113

**Suggested problems (do not hand in)**

1. With answers: 3.3, #1(a), 2(a), 3(ad), 7(a)
2. Without answers: 3.3 #1-7, 9
3. Handout 12

**Assignment, due Tuesday, November 22, via Canvas**:

1. Let $A, B$ and $C$ be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove or disprove each of the following.
   (a) If $g \circ f$ is an injection, then $g$ is an injection.
   (b) If $g \circ f$ is a surjection, then $f$ is a surjection.
   (c) If $g \circ f$ is a surjection, then $g$ is a surjection.
2. Let $A$ and $B$ be sets and let $f\colon A \to B$ and $g\colon B \to A$ be functions. Prove that if $g \circ f$ and $f \circ g$ are bijective, then so are $f$ and $g$.
3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $f$ and $g$ are both decreasing. Prove that $g \circ f$ is increasing.

# Assignment 11

**Thanksgiving break is Thursday, November 24 and Friday, November 25**;
There will no class these days.

Office hours are Wednesday, November 30, and the assignment will be due Thursday, December 1.

**Topics**: Inverse functions.

**Reading**: Section 3.3, p. 114-116

**Suggested problems (do not hand in)**

1. With answers: 3.3 #10(adgj), 11(a)
2. Without answers: 3.3 #10, 12, 14, 15, 17, 18, 19, 22
3. Handout 13

**Assignment, due Thursday, December 1, via Canvas**:

1. Define $f\colon \mathbf{R} - \{1\} \to \mathbf{R} - \{1\}$ by $f(x) = \frac{x+1}{x-1}$. Prove that $f$ is a bijection. Find a formula for the inverse $f^{-1}(x)$, and prove that it is correct.
2. Let $A, B$ and $C$ be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove that if $f$ and $g$ are invertible, then so is $g \circ f$, and prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^3 + x$. Prove that $f$ is invertible without finding a formula for $f^{-1}$.
4. Let $A$ and $B$ be sets and let $f\colon A \to B$ be a function. Suppose that $f$ has a *left inverse* $g$; that is, suppose that there exists a function $g\colon B \to A$ such that $g \circ f = id_A$. Prove that $f$ is injective.

# Assignment 12

**Topics**: Relations.

**Reading**: Section 4.2, p. 139-144 (stop at the proof of Theorem 4.2.6)

**Suggested problems (do not hand in)**

1. With answers: Section 4.2 #1(a), 3(ad), 4(a), 5(a), 12(a)
2. Without answers: Section 4.2 #1, 3, 4
3. Handout 14

**Assignment, due Tuesday, December 6, via canvas**:

1. Let $A = \{1, 2, 3\}$ and define a relation on $A$ by $a \sim b$ if $a + b \neq 3$. Determine whether this relation is reflexive, symmetric, transitive.
2. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a - b$ is divisible by 4.
   (a) Prove that this is an equivalence relation.
   (b) What integers are in the equivalence class of 18? (No proof necessary.)
   (c) What integers are in the equivalence class of 31? (No proof necessary.)
   (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)
3. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a^2 - b^2$ is divisible by 4.
   (a) Prove that this is an equivalence relation.
   (b) How many distinct equivalence classes are there? What are they? (No proof necessary.)
4. Let $A$ be a set, and let $P(A)$ be the power set of $A$. Assume that $A$ is not the empty set. Define a relation on $P(A)$ by $X \sim Y$ if $X \subseteq Y$. Is this relation reflexive, symmetric, and/or transitive? In each case give a proof, or disprove with a counterexample. (For a counterexample, give an example of $A$, $X$, and $Y$ that disproves the statement.)

# Final Exam

**Final exam** is Thursday December 8, 3:00pm - 5:30pm

The **last day of class** is Tuesday, December 6.

There will be **office hours** on Wednesday, December 7. I will send out a survey to find a time that works for everyone who is planning to attend.

The final exam will not be comprehensive, and will only cover content introduced after the midterm. Still, while I won't give you problems that are "just" about induction, contradiction, negations, etc. (so for example, I will not ask any irrationality questions) you will still need to use those techniques in some of your proofs.

The exam will be, roughly 8-10 questions, with multiple parts. Some questions will be "prove or disprove". For disproofs, please write out a counterexample as your disproof.

A typical exam will have one or two questions from each week of the course. You can expect a subset of the following:

- Images
- preimages
- Injectivity
- Surjectivity
- Compositions
- Invertibility
- Relations
- Problems from handouts 9-14